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Lie symmetries and infinite-dimensional Lie algebras of certain (1 + 1)-dimensional nonlinear evolution equations

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Abstract. In this paper we discuss the Lie symmetries, symmetry algebra and similarity reductions of two different equations introduced in the recent literature, namely, (i) a new coupled integrable dispersionless equation and (ii) a new coupled hyperbolic variational equation. We point out that both the systems admit, in contradistinction to conventional Lie algebras in $(1+1)$-dimensional systems, infinite-dimensional Lie algebras. Furthermore, we find physically interesting solutions for special choices of the symmetry parameters.

1. Introduction

One of the most powerful methods available to analyse nonlinear partial differential equations (PDEs) is the method of Lie groups [1–3]. An important feature of this method is that one can derive special solutions associated with nonlinear PDEs straightforwardly which are otherwise inaccessible through other methods [4–6]. The basic idea of the Lie group method is to seek the symmetry groups associated with a given differential equation under a continuous group of transformations and to find a reduction transformation from the symmetries. For PDEs, the reduction transformation can be used to reduce the number of independent variables by one; for example, a PDE with two independent variables to an ordinary differential equation (ODE). For a reduced ODE, one can check whether it is of Painlevé (P-) type or not and it is often the case that when the reduced ODE is of P-type it can be solved explicitly thereby leading to a solution of the original PDE [4].

In this direction some recent works have been devoted to the study of the symmetry groups of certain higher-dimensional nonlinear evolution equations in $(2+1)$ dimensions, which are generalizations of $(1+1)$-dimensional soliton equations, and it has been found that all these equations admit infinite-dimensional Lie point symmetry groups often with a specific Kac–Moody–Virasoro structure [7–11].

In contrast to the $(2+1)$-dimensional cases, solitons possessing nonlinear PDEs in $(1+1)$ dimensions generally admit only finite-dimensional point symmetry groups even though there are exceptions. As is already known, systems such as the Korteweg–de Vries (KdV), modified KdV, sine–Gordon, nonlinear Schrödinger, Heisenberg spin chain and so on [4] admit only finite-dimensional symmetry groups. However, there are other systems such as the Liouville equation, fourth-order shallow water equation [12] and the integrable dispersionless equation [13] which admit infinite-dimensional Lie point symmetry groups. It is possible that this list may include more examples. Of course the linear wave equation $u_{xy} = 0$ possesses an infinite-dimensional Lie algebra. Even in these cases the existence of

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Virasoro-type subalgebras has not been explored. Although the question is wide open as to whether integrability in $(1+1)$ dimensions is correlated to the structure of the Lie algebra of infinitesimal symmetries, it is of considerable interest to identify such nonlinear systems and analyse their symmetry algebras. In this paper we wish to point out the existence of infinite-dimensional Lie point symmetries in two other important physical systems in $(1+1)$ dimensions, namely a new coupled integrable dispersionless equation [14] and a new nonlinear hyperbolic variational equation [15], which arise in two different contexts, and explore the existence of infinite-dimensional Lie algebras in both systems. From the Lie symmetries we find similarity variables which in turn are used to reduce PDEs into ODEs. For the reduced ODEs we have analysed their integrability properties. We have also found interesting solutions for some particular cases.

The plan of this paper is as follows. In section 2 we investigate the Lie symmetries and similarity reductions associated with the new integrable dispersionless equation. We also find physically interesting solutions for special choices of symmetry parameters. In section 3 we investigate the invariance properties of the new integrable nonlinear hyperbolic variational equation. We find explicit solutions for some particular cases. In section 4 we present our conclusions.

2. Invariance analysis of the new coupled integrable dispersionless equation

2.1. Lie symmetries and similarity reductions

Recently, Konno and Oono have introduced a new coupled integrable dispersionless equation which is of the form [14, 16]

$$q_{xt} + (rs)_x = 0 \tag{2.1a}$$

$$r_{xt} - 2q_x r = 0 \tag{2.1b}$$

$$s_{xt} - 2q_x s = 0. \tag{2.1c}$$

By assuming that $r = s$ they have shown that the resultant system is solvable by the inverse scattering method. This special case has also been discussed by different authors in different contexts [17, 18]. However, quite recently the generalized form, equation (2.1), has been studied through the inverse scattering method and some remarkable soliton properties have been found [14, 19, 20]. In this work we clarify the invariance and integrability properties through the Lie group method.

The invariance of equation (2.1) under the one-parameter Lie group of infinitesimal point transformations,

$$x \longrightarrow X = x + \varepsilon\xi_1(t, x, q, r, s) \tag{2.2a}$$

$$t \longrightarrow T = t + \varepsilon\xi_2(t, x, q, r, s) \tag{2.2b}$$

$$q \longrightarrow Q = q + \varepsilon\phi_1(t, x, q, r, s) \tag{2.2c}$$

$$r \longrightarrow R = r + \varepsilon\phi_2(t, x, q, r, s) \tag{2.2d}$$

$$s \longrightarrow S = s + \varepsilon\phi_3(t, x, q, r, s) \qquad \varepsilon \ll 1 \tag{2.2e}$$

leads to the expressions (obtained by using the computer program MUMATH [21])

$$\xi_1 = f(x) \qquad \xi_2 = -(k_1 t + k_2) \qquad \phi_1 = k_1 q + g(t) \qquad \phi_2 = (2k_1 - k_3)r$$
$$\phi_3 = k_3 s \tag{2.3}$$

where k_1, k_2 and k_3 are arbitrary constants and $f(x)$ and $g(t)$ are arbitrary functions of x and t, respectively. Thus the system admits a set of infinite-dimensional Lie vector fields of the form

$$V = V_1(f) + V_2(g) + V_3 + V_4 + V_5 \tag{2.4}$$

where

$$V_1(f) = f(x)\frac{\partial}{\partial x} \tag{2.5a}$$

$$V_2(g) = g(t)\frac{\partial}{\partial q} \tag{2.5b}$$

$$V_3 = -t\frac{\partial}{\partial t} + q\frac{\partial}{\partial q} + 2r\frac{\partial}{\partial r} \tag{2.5c}$$

$$V_4 = -\frac{\partial}{\partial t} \tag{2.5d}$$

$$V_5 = -r\frac{\partial}{\partial r} + s\frac{\partial}{\partial s}. \tag{2.5e}$$

The *non-zero commutation relations* among the vector fields (2.5) are given by

$$[V_1(f_1), V_1(f_2)] = V_1(f_1 f_2' - f_2 f_1') \tag{2.6a}$$

$$[V_2, V_3] = V_2(g + t\dot{g}) \tag{2.6b}$$

$$[V_2, V_4] = V_2(t\dot{g}) \tag{2.6c}$$

where prime and dot denote differentiation with respect to x and t, respectively. It is interesting to note that, unlike the case of conventional Lie algebras of $(1+1)$-dimensional systems which are usually finite dimensional, the presence of arbitrary functions, $f(x)$ and $g(t)$ in the infinitesimal symmetries, leads to an infinite-dimensional Lie algebra. Furthermore, it is interesting to note that by restricting the arbitrary functions f and g to be Laurent polynomials we obtain a Kac–Moody–Virasoro-type subalgebra in the form

$$[V_1(x^m), V_1(x^n)] = (n-m)V_1(x^{n+m-1}) \tag{2.7a}$$

$$[V_2(t^m), V_2(t^n)] = 0. \tag{2.7b}$$

The similarity variables associated with the symmetries (2.3) can be obtained by solving the following characteristic equation:

$$\frac{\mathrm{d}x}{f(x)} = \frac{\mathrm{d}t}{-(k_1 t + k_2)} = \frac{\mathrm{d}q}{k_1 q + g(t)} = \frac{\mathrm{d}r}{(2k_1 - k_3)r} = \frac{\mathrm{d}s}{k_3 s}. \tag{2.8}$$

Solving equation (2.8) we obtain the following similarity variables,

$$z = \int^x \frac{\mathrm{d}x'}{f(x')} + \frac{1}{k_1}\log(k_1 t + k_2) \tag{2.9a}$$

$$F = (k_1 t + k_2)q + \int^t g(t')\,\mathrm{d}t' \tag{2.9b}$$

$$G = r(k_1 t + k_2)^p \tag{2.9c}$$

$$H = s(k_1 t + k_2)^{k_3/k_1} \tag{2.9d}$$

where $p = (2k_1 - k_3)/k_1$. Under this set of similarity transformations, equation (2.1) reduces to

$$F'' - k_1 F' + HG' + GH' = 0 \tag{2.10a}$$

$$G'' + (k_3 - 2k_1)G' - 2GF' = 0 \tag{2.10b}$$

$$H'' - k_3 H' - 2HF' = 0 \tag{2.10c}$$

where the prime stands for differentiation with respect to z.

In order to solve the nonlinear ODE (2.10) resulting from the similarity reduction, further investigations are necessary. As a first step, in order to check whether the reduced ODE is integrable or not, we first perform a P-analysis for equation (2.10).

2.2. Painlevé analysis

To begin with we represent the solution to equation (2.10) locally as a Laurent series and let the leading order be of the form

$$F = a_0\tau^p \qquad G = b_0\tau^q \qquad H = c_0\tau^r \qquad \tau = t - t_0 \to 0 \tag{2.11}$$

where t_0 is a movable singular point. Substituting (2.11) into (2.10) and equating the most singular terms we get $p = -1$, $q = -1$, $r = -1$ with the corresponding coefficient values as $a_0 = -1$ and $b_0c_0 = -1$, so that b_0 or c_0 is arbitrary.

By substituting

$$F = -\tau^{-1} + \sum_{n=1}^{\infty} a_n\tau^{n-1} \tag{2.12a}$$

$$G = b_0\tau^{-1} + \sum_{n=1}^{\infty} b_n\tau^{n-1} \tag{2.12b}$$

$$H = \frac{-1}{b_0}\tau^{-1} + \sum_{n=1}^{\infty} c_n\tau^{n-1} \tag{2.12c}$$

in equation (2.10) and equating the various powers of τ to zero we find that

$$a_1 = \text{arbitrary} \qquad b_1 = -\frac{k_3 - 2k_1}{2}b_0 \qquad c_1 = -\frac{k_3}{2b_0} \tag{2.13a}$$

$$a_2 = \text{arbitrary} \qquad b_2 = -b_0a_2 \qquad c_2 = \frac{a_2}{b_0} \tag{2.13b}$$

$$a_3 = 0 \qquad b_3 = \text{arbitrary} \qquad c_3 = \frac{b_3}{b_0^2} \tag{2.13c}$$

$$a_4 = \frac{2(b_4 - k_1b_3)}{3b_0} + \frac{a_2^2}{3} + \frac{k_3b_3}{3b_0} \qquad b_4 = \text{arbitrary} \qquad c_4 = -\frac{b_4 - k_1b_3}{b_0^2} \tag{2.13d}$$

and all the other higher-order coefficients are then determined uniquely in terms of the earlier coefficients.

Thus the Laurent series solution (2.12) is meromorphic and possesses a sufficient number of arbitrary constants (t_0, b_0, a_1, a_2, b_3 and b_4). Consequently, the ODE (2.10) possesses the P-property.

2.3. Soliton solutions

Even though it is very difficult to obtain the general solution associated with equation (2.10) we can obtain many interesting particular solutions by assuming special choices of the infinitesimal symmetries. For example, by choosing $f(x) = \text{constant}$, $g(t)$, k_1, $k_3 = 0$ and $k_2 = -1$ in (2.3) we get the travelling wave variable $z = x - ct$, $q = F(z)$, $r = G(z)$ and $s = H(z)$. Under these similarity transformations the reduced equation becomes

$$F'' - \frac{1}{c}HG' - \frac{1}{c}GH' = 0 \tag{2.14a}$$

$$G'' + \frac{2}{c} G F' = 0 \tag{2.14b}$$

$$H'' + \frac{2}{c} H F' = 0. \tag{2.14c}$$

Integrating equation (2.14a) once we obtain

$$F' = I_1 + \frac{HG}{c} \tag{2.15}$$

where I_1 is an integration constant. Substituting (2.15) in (2.14b) and (2.14c) we get

$$G'' + \frac{2I_1}{c} G + \frac{2HG^2}{c^2} = 0 \tag{2.16a}$$

$$H'' + \frac{2I_1}{c} H + \frac{2GH^2}{c^2} = 0. \tag{2.16b}$$

Obviously the system (2.16) admits an integral

$$HG' - GH' = I_2 \tag{2.17}$$

where I_2 is a second integration constant. Using equation (2.17) in equation (2.16a), we can eliminate one variable H, and the resultant equation takes the form

$$GG''' - 3G'G'' - \frac{4I_1}{c} GG' - \frac{2}{c^2} I_2 G^2 = 0. \tag{2.18}$$

For $I_2 = 0$ equation (2.18) admits an integral of the form

$$GG'' - 2G'^2 - \frac{2I_1}{c} G^2 = I_3 \tag{2.19}$$

where I_3 is the third integration constant. The first integral associated with the above equation can be written as [22]

$$G'^2 = I_4 G^4 - \frac{2I_1}{c} G^2 - \frac{I_3}{2} \tag{2.20}$$

where I_4 is the fourth integration constant. The general solution of equation (2.20) can be expressed in terms of elliptic functions, a special case of which is

$$G = \sqrt{\frac{2I_1}{cI_4}}\,\mathrm{sech}\left[\sqrt{\frac{-2I_1}{c}} z + \delta\right] \tag{2.21a}$$

$$H = -\sqrt{2I_1 I_4 c^3}\,\mathrm{sech}\left[\sqrt{\frac{-2I_1}{c}} z + \delta\right] \tag{2.21b}$$

$$F' = I_1\left[1 - 2c\,\mathrm{sech}^2\left[\sqrt{\frac{-2I_1}{c}} z + \delta\right]\right] \tag{2.21c}$$

provided $-2I_1/c$ is positive and I_3 is zero. Rewriting the variables $z = x - ct$, $F = q$, $G = r$ and $H = s$ we get the bright and dark soliton-type solution derived by Konno and Kakuhata [14].

2.4. Subcases

In addition to the above travelling wave solution one can also look for other particular solutions of equation (2.10) that are invariant under the subgroup of the symmetry groups which correspond to the Lie algebras (2.5). In the following we consider the following *non-trivial cases* only.

Case (i). $k_1, k_2, k_3, g(t) \neq 0$ *and* $f(x) = 0$
The similarity variables are

$$z = x \qquad F = (k_1 t + k_2)q + \int^t g(t')\,\mathrm{d}t' \qquad G = r(k_1 t + k_2)^{(2k_1 - k_3)/k_1}$$
$$H = s(k_1 t + k_2)^{k_3/k_1}. \tag{2.22}$$

Under this similarity transformation the reduced equation takes the form

$$F' - \frac{GH'}{k_1} - \frac{HG'}{k_1} = 0 \tag{2.23a}$$
$$G' - \frac{2}{(k_3 - 2k_1)} F'G = 0 \tag{2.23b}$$
$$H' + \frac{2}{k_3} F'H = 0. \tag{2.23c}$$

Integrating equation (2.23*a*), we get

$$F = \frac{GH}{k_1} + I_1 \tag{2.24}$$

where I_1 is an integration constant. Now integrating equation (2.23*c*), we get

$$F = \log \frac{I_2}{H^{k_3/2}} \tag{2.25}$$

where I_2 is a second integration constant. Solving equations (2.24) and (2.25) we obtain

$$G = \frac{k_1}{H} \log \frac{I_2}{H^{k_3/2}} - \frac{I_1 k_1}{H}. \tag{2.26}$$

Substituting (2.26) and (2.25) in (2.23*b*) we find a trivial solution $H = \text{constant} = I_3$ which in turn leads to the following solution for (2.1),

$$q = \frac{-1}{(k_1 t + k_2)} \int^t g(t')\,\mathrm{d}t' + \frac{c}{(k_1 t + k_2)} \tag{2.27a}$$
$$r = \left[\frac{k_1}{I_3} c - \frac{I_1 k_1}{k_3}\right] (k_1 t + k_2)^{(k_3 - 2k_1)/k_1} \tag{2.27b}$$
$$s = \frac{I_3}{(k_1 t + k_3)^{k_3/k_1}} \tag{2.27c}$$

where $c = \log(I_2/H^{K_3/2})$, through (2.25), (2.26) and (2.22).

Case (ii). $k_3, f(x), g(t) \neq 0$ *and* $k_1, k_2 = 0$
The similarity variables lead to the invariant solution

$$q = I_1 \int^x \frac{\mathrm{d}x'}{f(x')} + p(t) \tag{2.28a}$$
$$r = I_2 \exp\left[\frac{-2I_1 t}{k_3} - k_3 \int^x \frac{\mathrm{d}x'}{f(x')}\right] \tag{2.28b}$$
$$s = I_3 \exp\left[\frac{2I_1 t}{k_3} + k_3 \int^x \frac{\mathrm{d}x'}{f(x')}\right] \tag{2.28c}$$

where I_1, I_2 and I_3 are integration constants and $p(t)$ is an arbitrary function of t.

Case (iii). $k_1, k_2, g(t) \neq 0$ *and* $k_3, f(x) = 0$

The similarity reduction leads to the particular solution of the form

$$q = \frac{-1}{(k_1 t + k_2)} \int^t g(t')\,\mathrm{d}t' + \frac{I_1}{(k_1 t + k_2)} \qquad r = \frac{I_2}{(k_1 t + k_2)^2} \qquad s = I_3. \tag{2.29}$$

Case (iv). $k_2, k_3, f(x) \neq 0$ *and* $k_1, g(t) = 0$

The similarity variables are

$$z = -k_2 \int^x \frac{\mathrm{d}x'}{f(x')} - t \qquad F = q \qquad G = r \exp\left[\frac{-k_3 t}{k_2}\right] \qquad H = s \exp\left[\frac{k_3 t}{k_2}\right]. \tag{2.30}$$

The reduced ODE takes the form

$$F'' + GH' + HG' = 0 \tag{2.31a}$$

$$G'' - \frac{k_3}{k_2} G' + 2k_2 F' G = 0 \tag{2.31b}$$

$$H'' + \frac{k_3}{k_2} H' + 2k_2 F' H = 0. \tag{2.31c}$$

Equations (2.31) can be integrated to obtain a third-order nonlinear ODE in G or H. Although it is difficult to integrate it further, one can show that it satisfies the Painlevé property.

Case (v). $f(x), g(t) \neq 0$ *and* $k_1, k_2, k_3 = 0$

In this case we obtain a particular solution of the form

$$q = f(t) \qquad r = g(t) \qquad s = h(t) \tag{2.32}$$

where $f(t)$, $g(t)$ and $h(t)$ are arbitrary functions of t.

Case (vi). $k_1, k_3, g(t) \neq 0$ *and* $k_2, f(x) = 0$

The similarity variables are

$$z = x \qquad F = qt + \frac{1}{k_1} \int^t g(t')\,\mathrm{d}t' \qquad G = rt^{(2k_1 - k_3)/k_1} \qquad H = st^{k_3/k_1}. \tag{2.33}$$

The reduced equation takes the form

$$F' - GH' - HG' = 0 \tag{2.34a}$$

$$G' - \frac{2k_1}{(k_3 - 2k_1)} F' G = 0 \tag{2.34b}$$

$$H' + \frac{2k_1}{k_3} F' H = 0. \tag{2.34c}$$

By following the steps given in case (i) one can obtain the following solution of (2.34) as

$$q = \frac{c}{t} - \frac{1}{k_1 t} \int^t g(t')\,\mathrm{d}t' \tag{2.35a}$$

$$r = \left(\frac{c}{I_3} - I_1 I_3\right) t^{(k_3 - 2k_1)/k_1} \tag{2.35b}$$

$$s = I_3 t^{-k_3/k_1} \tag{2.35c}$$

where I_1, I_2, I_3 are constants and $c = \log(I_2/I_3^{k_3/2k_1})$.

Case (vii). $k_1, f(x) = 0$
In this case we obtain a particular solution of the form

$$q = -\frac{1}{k_2}\int^t g(t')\,\mathrm{d}t' + \log G^{k_3 I_2/2k_2} \tag{2.36a}$$

$$r = G(x)\exp\left[\frac{k_3 t}{k_2}\right] \tag{2.36b}$$

$$s = \frac{I_1}{G(x)}\exp\left[-\frac{k_3 t}{k_2}\right] \tag{2.36c}$$

where $g(t)$ and $G(x)$ are arbitrary functions of t and x, respectively, and I_1 and I_2 are constants.

3. Invariance analysis of a new coupled hyperbolic variational equation

Recently Hunter and Zheng have investigated the nonlinear PDE [15]

$$u_{xxt} + 2u_x u_{xx} + u u_{xxx} = 0 \tag{3.1}$$

and shown that equation (3.1) is a completely integrable, bi-variational and bi-Hamiltonian system. This equation arises in two different physical contexts. It describes the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal director fixed and it is also the high-frequency limit of the Camassa–Holm (CH) equation [23]

$$u_t + 2\kappa u_x + 3u u_x = u_{xxt} + 2u_x u_{xx} + u u_{xxx} \tag{3.2}$$

which is an integrable model equation for shallow water waves. Earlier [24] we investigated the invariance properties of the CH equation and proved that the CH equation admits a three-parameter symmetry group with the infinitesimals

$$\xi_1 = a_1\kappa t + a_2 \qquad \xi_2 = a_1 t + b_1$$

and

$$\phi = -a_1(\kappa + u) \tag{3.3}$$

where a_1, a_2 and b_1 are arbitrary constants and κ is the system parameter. Now we wish to show that the limiting case of the CH equation admits an infinite-dimensional Lie algebra.

The invariance of equation (3.1) under the infinitesimal transformation

$$x \longrightarrow X = x + \varepsilon\xi_1(t, x, u) \tag{3.4a}$$

$$t \longrightarrow T = t + \varepsilon\xi_2(t, x, u) \tag{3.4b}$$

$$u \longrightarrow U = u + \varepsilon\phi(t, x, u) \tag{3.4c}$$

leads to the expressions

$$\xi_1 = x\dot{f}(t) + k_1 x + g(t) \tag{3.5a}$$

$$\xi_2 = f(t) \tag{3.5b}$$

$$\phi = k_1 u + x\ddot{f}(t) + \dot{g}(t) \tag{3.5c}$$

where k_1 is an arbitrary constant, $f(t)$ and $g(t)$ are arbitrary functions of t and a dot denotes differentiation with respect to the variable t. The general element of the Lie algebra can be written as

$$V = V_1(f) + V_2(g) + V_3 \tag{3.6}$$

where f and g are arbitrary functions of t and

$$V_1(f) = x\dot{f}(t)\frac{\partial}{\partial x} + f(t)\frac{\partial}{\partial t} + x\ddot{f}(t)\frac{\partial}{\partial u} \tag{3.7a}$$

$$V_2(g) = g(t)\frac{\partial}{\partial x} + \dot{g}(t)\frac{\partial}{\partial u} \tag{3.7b}$$

$$V_3 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}. \tag{3.7c}$$

The commutation relations between the Lie vector fields lead to an infinite-dimensional Lie algebra of the form

$$[V_1(f_1), V_1(f_2)] = V_1(f_1\dot{f}_2 - f_2\dot{f}_1) \tag{3.8a}$$

$$[V_2(g_1), V_2(g_2)] = 0 \tag{3.8b}$$

$$[V_1(f), V_2(g)] = V_2(f\dot{g} - g\dot{f}) \tag{3.8c}$$

$$[V_1, V_3] = 0 \tag{3.8d}$$

$$[V_2, V_3] = V_2. \tag{3.8e}$$

By restricting the arbitrary functions f and g to Laurent polynomials again, we obtain a Virasoro–Kac–Moody-type subalgebra of the form

$$[V_1(t^m), V_1(t^n)] = (n-m)V_1(t^{n+m-1}) \tag{3.9a}$$

$$[V_2(t^m), V_2(t^n)] = 0 \tag{3.9b}$$

$$[V_1(t^m), V_2(t^n)] = (n-m)V_2(t^{n+m-1}). \tag{3.9c}$$

The similarity variables associated with the infinitesimal symmetries (3.5) can be found by solving the characteristic equation. For the present case they turn out to be

$$z = \frac{x}{f(t)}\exp\left[-\int^t \frac{k_1}{f(t')}\,\mathrm{d}t'\right] - \int^{t'} \frac{g(t')}{f^2(t')}\exp\left[-\int^t \frac{k_1\,\mathrm{d}t''}{f(t'')}\right]\mathrm{d}t' \tag{3.10a}$$

$$\begin{aligned} F = {} & \left(u - \frac{g(t)}{f(t)}\right)\exp\left[-\int^t \frac{k_1}{f(t')}\,\mathrm{d}t'\right] \\ & -(\dot{f}+k_1)\int^t \frac{g(t')}{f^2(t')}\exp\left[-\int^{t'} \frac{k_1}{f(t'')}\,\mathrm{d}t''\right]\mathrm{d}t' - z\dot{f}. \end{aligned} \tag{3.10b}$$

Under this set of similarity variables the PDE (3.1) can be written as

$$(F - k_1 z)F''' + 2F'F'' - k_1 F'' = 0. \tag{3.11}$$

Integrating equation (3.11) once, we find

$$FF'' - zF'' + \tfrac{1}{2}F'^2 = I_1 \tag{3.12}$$

where I_1 is an integration constant. Rewriting equation (3.12), we get

$$(z - F)F'' = \frac{\hat{I}_1 + F'^2}{2}. \tag{3.13}$$

Integrating equation (3.13) again, we obtain [25]

$$(z - F)(F'^2 + \hat{I}_1)\exp\left[-2\tan^{-1}\left(\frac{F'}{\hat{I}_1}\right)\right] = I_2 \tag{3.14}$$

where I_2 is a second integration constant and $\hat{I}_1 = -(I_1/2)$. Integrating equation (3.14) we can obtain the solution for the ODE (3.11).

3.1. Special group invariant solutions

It is interesting to note that in addition to the above general symmetry reduction one can also consider special cases.

Case (i). The invariance of equation (3.1) under the infinitesimal symmetries $f(t)$ and $g(t)$ alone ($k_1 = 0$) leads to the similarity variables

$$z = \frac{x}{f(t)} - \int^t \frac{g(t')\,\mathrm{d}t'}{f^2(t')} \tag{3.15a}$$

$$F = u - \dot{f}\int^t \frac{g(t')\,\mathrm{d}t'}{f^2(t')} - \frac{g(t)}{f(t)} - z\dot{f}. \tag{3.15b}$$

The corresponding ODE turns out to be

$$FF''' + 2F'F'' = 0. \tag{3.16}$$

Integrating equation (3.16) twice we get

$$FF'^2 = I_1F + I_2 \tag{3.17}$$

where I_1 and I_2 are integration constants. Choosing $I_1 = -1$ we can write the solution as [25]

$$F' = \cot u \qquad F = I_2 \sin^2 u \qquad z = I_2(u - \sin u \cos u) + I_3 \tag{3.18}$$

where z is given in equation (3.15).

Case (ii). Similarly the invariance of the equation (3.11) under the infinitesimal symmetries $f(t)$ and k_1 alone ($g(t) = 0$) leads to the ODE

$$(F - k_1z)F''' + 2F'F'' - k_1F'' = 0 \tag{3.19}$$

which is the same as that of equation (3.11).

Case (iii). Finally the invariance of equation (3.1) under the infinitesimal symmetries $g(t)$ and k_1 alone ($f(t) = 0$) leads to the particular solution for equation (3.1) of the form

$$u = \frac{-\dot{g}(t)}{k_1} + w(t)(c_1x + g(t)) \tag{3.20}$$

where w and g are arbitrary functions of t.

Case (iv). The similarity reduction under the possibility $k_1,\ f(t) = 0$ ($g(t) \neq 0$) leads to the particular solution of the form

$$u = w(t) + \frac{\dot{g}(t)}{g(t)}x \tag{3.21}$$

where w is an arbitrary function of t. This exhausts all possible similarity reductions.

4. Conclusions

In this paper, we have pointed out that the two new completely integrable PDEs, namely, the new coupled integrable dispersionless equations and a nonlinear hyperbolic variational equation, admit infinite-dimensional Lie algebras which is not a common occurrence in $(1 + 1)$ dimensions. This enables us to obtain a class of interesting solutions to these equations. The connection between these infinite-dimensional Lie algebras and the integrability is a further interesting problem to study. Furthermore, it is also of interest to study the non-classical symmetries associated with these systems, which we are pursuing at present.

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